# Étale base change of the saturated de Rham-Witt complex

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## 1 Introduction

Let $R$ be a commutative $\mathbb{F}_p$-algebra, so that we can form its saturated de Rham-Witt complex $\mathcal{W}\Omega^*_R$. Last time we saw that for any localization $R[s^{-1}]$, there is a natural description of $\mathcal{W}\Omega^*_{R[s^{-1}]}$ in terms of $\mathcal{W}\Omega^*_R$; namely, we have

$$\mathcal{W}\Omega^*_{R[s^{-1}]} \simeq \mathcal{W}(\mathcal{W}\Omega^*_R[[s]^{-1}]). \tag{1}$$

Moreover, this allowed us to promote $\mathcal{W}\Omega^*$ to a sheaf of abelian groups (equipped with the structure of a Dieudonné algebra) on an arbitrary $\mathbb{F}_p$-scheme.

In this talk, we will repeat this process for the étale topology instead of the Zariski topology. Namely, if $S$ is an étale $R$-algebra, we will relate $\mathcal{W}\Omega^*_S$ to $\mathcal{W}\Omega^*_R$, and we will promote $\mathcal{W}\Omega^*$ to a sheaf of abelian groups on the étale site of an $\mathbb{F}_p$-scheme.

This will[1] be useful at the end of the paper, when we study the crystalline comparison of BMS's $A\Omega$-cohomology theory. The key technical tool there will be to work locally on "small affine opens" that are étale covers of tori.

## 2 Preparations

We begin by stating a couple of facts that will be used repeatedly through the talk.

**Theorem 2.1.** *(5.4.1) Let $f : A \to B$ be an étale morphism of $\mathbb{F}_p$-algebras. Then for each $n$, the induced map $W_n(A) \to W_n(B)$ is also étale, and*

$$\begin{array}{ccc} W_n(A) & \longrightarrow & W_n(B) \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R} \\ W_{n-1}(A) & \longrightarrow & W_{n-1}(B) \end{array}$$

[*]Notes for a talk in Berkeley's number theory seminar, on Bhatt-Lurie-Mathew's paper *Revisiting the de Rham-Witt complex*.

[1]At least, it should. At present I can't find any explicit references to this later.

*is a pushout square.*

This was proved by Illusie in 1979, and there are various strengthenings by various people that weaken the hypothesis that everything is over $\mathbb{F}_p$. We omit the proof.

A related theorem, whose proof we also omit, is as follows:

Theorem: The functor $W_n$ is an equivalence between the categories of étale $R$-modules and étale $W_n(R)$-modules.

Remark 5.4.2: Let $f : A \to B$ be an étale morphism of $\mathbb{F}_p$-algebras. Then for all $n, k \geq 0$, the diagram

$$\begin{array}{ccc} W_n(A) & \xrightarrow{W_n(f)} & W_n(B) \\ \downarrow{\scriptstyle F^k} & & \downarrow{\scriptstyle F^k} \\ W_n(A) & \xrightarrow{W_n(f)} & W_n(B) \end{array}$$

is a pushout square of commutative rings.

*Proof.* We must prove that the induced map

$$\theta : W_n(A) \otimes_{F^k, W_n(A)} W_n(B) \to W_n(B) \tag{2}$$

is an isomorphism. This is a map of étale $W_n(A)$-algebras, and isomorphisms of étale algebras can be detected on the special fiber. So we base change along $W_n(A) \to A$, which reduces to the case $n = 1$. Even this case is nontrivial, but we saw a proof of it last month. □

# 3 Setup and statements of theorems

**Definition 3.1.** Let $f : A^* \to B^*$ be a map of cdgas. We say $f$ is *étale* if $f : A^0 \to B^0$ is étale and $f$ induces an isomorphism of graded rings $A^* \otimes_{A^0} B^0 \to B^*$.

**Proposition 3.2.** *Let $A^*$ be a cdga. Then:*

1. *For every étale $A^0$-algebra $R$, there exists an étale $A^*$-algebra $B^*$ and an isomorphism of $A^0$-algebras $R \simeq B^0$.*

2. *Let $B^*$ be an étale $A^*$-algebra. Then for any $A^*$-algebra $C^*$, the natural map*

$$\mathrm{Hom}_{A^*}(B^*, C^*) \to \mathrm{Hom}_{A^0}(B^0, C^0)$$

*is bijective.*

3. *The functor $B^* \mapsto B^0$ is an equivalence from the category of étale $A^*$-algebras to the category of étale $A^0$-algebras.*

*Proof.* Claim (3) is just a restatement of (1) and (2): a functor is an equivalence of categories if and only if it is essentially surjective and fully faithful; these are respectively the contents of the first two claims.

Some motivation for proving (3): we would like to just say that the inverse functor is $R \mapsto A^* \otimes_{A^0} R$. The problem with this is that $A^* \otimes_{A^0} R$ doesn't come with a differential in general. (If you try to write it down as a fibered product of cdgas, you'll realize that $A^0 \to A^*$ isn't a cdga map.) To fix this, we need to use the fact that $A^0 \to R$ is étale, as follows.

To prove (1), let $R$ be étale over $A^0$. The cotangent exact sequence gives an isomorphism $R \otimes_{A^0} \Omega^1_{A^0} \to \Omega^1_R$. It follows by taking exterior algebras that $\Omega^*_R = R \otimes_{A^0} \Omega^*_{A^0}$ as a graded $R$-algebra. Tensoring over $\Omega^*_{A^0}$ with $A^*$ gives an isomorphism of graded algebras $\Omega^*_R \otimes_{\Omega^*_{A^0}} A^* \to R \otimes_{A^0} A^*$. The left side was constructed a cdga, so the right side inherits this structure. This completes the proof of (1). To prove (2), one needs to check that this construction has the right universal property (TO DO: and that it gives the unique $A^*$-cdga structure on $A^* \otimes_{A^0} R$, right?); we omit this. □

**Definition 3.3.** Let $f : A^* \to B^*$ be a morphism of strict Dieudonné algebras. We say $f$ is *$V$-adically étale* if, for each $n$, $\mathcal{W}_n(f) : \mathcal{W}_n(A)^* \to \mathcal{W}_n(B)^*$ is étale in the sense defined earlier.

**Theorem 3.4.** *(5.3.4) Let $A^*$ be a strict Dieudonné algebra. Then:*

1. *For every étale $A^0/VA^0$-algebra $S$,[2] there is a $V$-adically étale morphism of strict Dieudonné algebras $A^* \to B^*$ realizing $S$ as $B^0/VB^0$.*

2. *Let $f : A^* \to B^*$ be a $V$-adically étale morphism of strict Dieudonné algebras. Then, for every morphism $A^* \to C^*$ of strict Dieudonné algebras, the natural map*

$$\mathrm{Hom}_{\mathrm{DC}_{A^*}}(B^*, C^*) \to \mathrm{Hom}_{A^0/VA^0}(B^0/VB^0, C^0/VC^0)$$

   *is bijective.*

3. *The functor $B^* \mapsto B^0/VB^0$ is an equivalence from the category of $V$-adically étale strict Dieudonné algebras over $A^*$ to the category of étale $A^0/VA^0$-algebras.*

The proof takes some work, and we will postpone it until later. For now, we will see how we can use it to understand the behavior of $\mathcal{W}\Omega^*$ when we pass to an étale cover. The following corollary is analogous to our description of $\mathcal{W}\Omega^*_{R[s^{-1}]}$ in terms of $\mathcal{W}\Omega^*_R$:

**Corollary 3.5.** *(5.3.5) Let $R \to S$ be an étale map of $\mathbb{F}_p$-algebras. Then $\mathcal{W}\Omega^*_R \to \mathcal{W}\Omega^*_S$ is $V$-adically étale, and for each $n$ it induces an isomorphism*

$$\mathcal{W}_n\Omega^*_R \otimes_{W_n(R)} W_n(S) \to \mathcal{W}_n\Omega^*_S.$$

*Proof.* Let $A^*$ be a saturated de Rham-Witt complex of $R$; in particular, this comes with a ring map $R \to A^0/VA^0$ satisfying a universal property. Set $S' = S \otimes_R (A^0/VA^0)$; this is an

[2] I have changed BLM's notation slightly here to be more consistent with the notation used in their proof.

étale $A^0/VA^0$-algebra. By the theorem, it corresponds to a (unique) $V$-adically étale strict Dieudonné algebra $B^*$ over $A^*$, with $B^0/VB^0 \cong S'$. Tracing through the universal properties, we can see that the map $S \to S' = B^0/VB^0$ exhibits $B^*$ as a saturated de Rham-Witt complex of $S$.

For the second part, for each $n \geq 0$, we have maps

$$\mathcal{W}_n(A)^* \otimes_{W_n(R)} W_n(S) \xrightarrow{\alpha} \mathcal{W}_n(A)^* \otimes_{W_n(A^0/VA^0)} W_n(S') \xrightarrow{\beta} \mathcal{W}_n(B)^*. \tag{3}$$

Since $B^*$ is $V$-adically étale over $A^*$, we have by definition that $\beta$ is an isomorphism. To see that $\alpha$ is an isomorphism, rewrite it as

$$\begin{aligned} \mathcal{W}_n(A)^* \otimes_{W_n(R)} W_n(S) &= \mathcal{W}_n(A)^* \otimes_{W_n(A^0/VA^0)} W_n(A^0/VA^0) \otimes_{W_n(R)} W_n(S) && (4)\\ &\to \mathcal{W}_n(A)^* \otimes_{W_n(A^0/VA^0)} W_n(S'), && (5) \end{aligned}$$

and use the fact that $W_n(A^0/VA^0) \otimes_{W_n(R)} W_n(S) \to W_n(S')$ is an isomorphism. □

We are now ready to state the étale sheaf property, and to prove it conditionally on the theorem. For a scheme $X$, let $\mathcal{U}_{\text{aff,ét}}(X)$ denote the category of étale maps $U \to X$ such that $U$ is affine. If $X$ is an $\mathbb{F}_p$-scheme, we then get a presheaf of Dieudonné algebras $\mathcal{W}\Omega^*_{X_{\text{ét}}}$ on $\mathcal{U}_{\text{aff,ét}}$:

$$\mathcal{W}\Omega^*_{X_{\text{ét}}}(U \to X) = \mathcal{W}\Omega^*_{\mathscr{O}_U(U)}. \tag{6}$$

**Theorem 3.6.** *(5.3.7) Let $X$ be an $\mathbb{F}_p$-scheme. The presheaf $\mathcal{W}\Omega^*_{X_{\acute{e}t}}$ is a sheaf on the étale topology of $\mathcal{U}_{\textit{aff,ét}}$.*

*Proof.* As in the Zariski analogue of this, we assume without loss of generality that $X = \operatorname{Spec} R$ is affine, and show that each $\mathcal{W}_n\Omega^i$ is a sheaf. Then the category of étale $R$-algebras is equivalent to the category of étale $W_n(R)$-algebras via $S \mapsto W_n(S)$, so we can regard $\mathcal{U}_{\text{aff,ét}}(\operatorname{Spec} R)$ instead as $\mathcal{U}_{\text{aff,ét}}(\operatorname{Spec} W_n(R))$. Under this identification, the presheaf $\mathcal{W}_n\Omega^i_{X_{\text{ét}}}$ becomes the quasicoherent étale sheaf $\widetilde{M}_{\text{ét}}$, where $M$ is the $W_n(R)$-module $\mathcal{W}_n\Omega^i_R$, by Corollary 5.3.5 from earlier. □

# 4 Proof of Theorem 5.3.4

We now begin the proof of Theorem 5.3.4.

*Proof.* As before, property (3) is equivalent to properties (1) and (2). We will sketch the proofs of (2) and then (1).

For (2), fix a morphism $\overline{f} : B^0/VB^0 \to C^0/VC^0$ of $A^0/VA^0$-algebras. We claim that $\overline{f}$ can be lifted to a unique morphism $f : B^* \to C^*$ of strict Dieudonné $A^*$-algebras. For each $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccccccc} \mathcal{W}_n(A)^0 & = & A^0/V^nA^0 & \longrightarrow & C^0/V^nC^0 & = & \mathcal{W}_n(C)^0 \\ & & \downarrow & \overset{f_n^0}{\nearrow} & \downarrow & & \\ \mathcal{W}_n(B)^0 & = & B^0/V^nB^0 & \xrightarrow[\overline{f}\circ R^{n-1}]{} & C^0/VC^0. & & \end{array}$$

The map on the left is étale by assumption, and the map on the right is a nilpotent thickening, so there is a unique map $f_n^0$ filling in the diagonal. By Proposition 5.3.2, each $f_n^0$ can be extended to a unique map $f_n : \mathcal{W}_n(B)^* \to \mathcal{W}_n(C)^*$ of $\mathcal{W}_n(A)^*$-algebras. These maps are compatible, so passing to the limit gives a map $f : B^* \to C^*$ of cdgas over $A^*$. To complete the proof, one needs to verify that $f$ is compatible with Frobenius; we omit this.

For (1), suppose we are given a strict Dieudonné algebra $A^*$, with $R := A^0/VA^0$, and an étale $R$-algebra $S$. We want to produce a strict Dieudonné algebra $B^*$ with $B^0/VB^0 \cong S$ as an $R$-algebra. We will construct this at finite levels, as a strict Dieudonné tower; it will then remain to check the Dieudonné tower and Dieudonné algebra axioms.

Since $A^0 = W(R)$, we have $\mathcal{W}_n(A)^0 = W(R)/V^nW(R) = W_n(R)$, and $W_n(S)$ is étale over this by Theorem 5.4.1. By part (1) of Proposition 5.3.2, there exists a cdga $B_n^*$, étale over $\mathcal{W}_n(A)^*$, with $B_n^0 \cong W_n(S)$ as $\mathcal{W}_n(A)^0$-algebras. (Explicitly, $B_n^* = \mathcal{W}_n(A)^* \otimes_{W_n(R)} W_n(S)$.) By part (2) of Proposition 5.3.2, the Frobenius and restriction maps $R, F : W_n(S) \to W_{n-1}(S)$ extend uniquely to cdga maps $R, F : B_n^* \to B_{n-1}^*$. (In appealing to the proposition here, $\mathcal{W}_n(A)^*$ plays the role of $A^*$, $B_n^*$ that of $B^*$, and $B_{n-1}^*$ that of $C^*$.)

At this point, we can define $B^* = \lim_{\leftarrow n} B_n^*$; this inherits a Frobenius map. Unfortunately, we need a different method to get Verschiebung maps $B_n^* \to B_{n+1}^*$, since the Verschiebung isn't (and can't be) an algebra homomorphism.

To get the Verschiebung, we will lie a little, and correct the error later. We already have Verschiebung maps $V : \mathcal{W}_n(A)^* \to \mathcal{W}_{n+1}(A)^*$, so we base change along $W_n(R) \to W_n(S)$ to get $V : B_n^* \to B_{n+1}^*$. This is wrong because the Verschiebung isn't $W_n(R)$-linear; to fix it, we will need to develop a little theory of Frobenius-semilinear maps between modules over $W_n(R)$.

Once we have constructed $V$, axioms (1)-(5) of strict Dieudonné towers are trivial, as are Dieudonné algebra axioms (1)-(3). To verify (6)-(8), we will tell some more small lies. Properties (6) (saturation), (7) ("$\ker R = p$-torsion"), and (8) ("$\ker R_n = \operatorname{im} V^n + \operatorname{im} dV^n$") say respectively that the three sequences

$$B_{n+1}^* \xrightarrow{F} B_n^* \xrightarrow{d} B_n^{*+1}/pB_n^{*+1} \tag{7}$$

$$B_{n+1}^*[p] \xrightarrow{\text{id}} B_{n+1}^* \xrightarrow{R} B_n^* \tag{8}$$

$$B_1^* \oplus B_1^{*-1} \xrightarrow{(V^n, dV^n)} B_{n+1}^* \xrightarrow{R} B_n^* \tag{9}$$

are exact. These come from base-changing the corresponding sequences for $\mathcal{W}_n(A)^*$ along the flat map $W_n(R) \to W_n(S)$. Again, the problem is that $F$ and $V$ aren't $W_n$-linear. (The argument works as is for (7).) □

# 5 Tying up loose ends

Definition 5.4.3: Let $A$ be an $\mathbb{F}_p$-algebra and $M$ a $W(A)$-module. We say $M$ is *nilpotent* if $M$ is annihilated by $V^nW(A)$ for some $n$; i.e. $M$ is a $W_n(A)$-module for some $n$.

Definition 5.4.4: Let $A \to B$ be an étale map of $\mathbb{F}_p$-algebras. Suppose $M$ is a nilpotent $W(A)$-module, so that $M$ is also a $W_n(A)$-module for some $n$. We let $M_B$ denote the nilpotent $W(B)$-module $M \otimes_{W_n(A)} W_n(B)$. By the pushout theorem, this is independent of $n$.

Remark 5.4.5: The functor $M \mapsto M_B$ is left adjoint to the forgetful functor from nilpotent $W(B)$-modules to nilpotent $W(A)$-modules. It is also exact, as $W_n(A) \to W_n(B)$ is étale and therefore flat.

Definition 5.4.6: If $M$ is a $W(A)$-module, we let $M_{(k)}$ denote its restriction of scalars along $F^k : W(A) \to W(A)$. That is, $M_{(k)}$ is the same as $M$ as an abelian group, but its $W(A)$-module structure is twisted by the $k$-fold Frobenius.

Remark: Frobenius twists preserve nilpotence of $W(A)$-modules: if $M$ is a $W_n(A)$-module, when $M_{(k)}$ is as well. Moreover, they commute with étale base change: given an étale map $A \to B$, if $A \to B$ is an étale map of $\mathbb{F}_p$ algebras and $M$ is a nilpotent $W(A)$-module, then $(M_B)_{(k)} = (M_{(k)})_B$. This follows immediately from Remark 5.4.2, once we have chosen an $n$ such that $M$ is a $W_n(A)$-module.

Definition 5.4.9: Let $M$ and $N$ be nilpotent $W(A)$-modules. We say a map of abelian groups $f : M \to N$ is $(F^k, F^\ell)$-linear if it satisfies the identity

$$f((F^k x)m) = (F^\ell x) f(m) \in N \tag{10}$$

for all $x \in W(A)$ and $m \in M$. Equivalently, such $f$ is an honest map of $W(A)$-modules $M_{(k)} \to N_{(\ell)}$.

Corollary 5.4.10: Let $A \to B$ be étale as before. Let $M$ be a nilpotent $W(A)$-module and $N$ and nilpotent $W(B)$-module, viewed as a nilpotent $W(A)$-module by restriction of scalars. Then any $(F^k, F^\ell)$-linear map $f : M \to N$ extends uniquely to an $(F^k, F^\ell)$-linear map $M_B \to N$.

*Proof.* We are given a linear map $f : M_{(k)} \to N_{(\ell)}$, which factors uniquely through $(M_{(k)})_B = (M_B)_{(k)}$. □

Remark 5.4.11: Let $A, B$ be as above. Let $M, N$, and $P$ be nilpotent $W(A)$-modules, with maps $f : M \to N$ and $g : N \to P$ that are respectively $(F^a, F^b)$-linear and $(F^c, F^d)$-linear. If $M \to N \to P$ is an exact sequence of abelian groups, then $M_B \to N_B \to P_B$ is too.

*Proof.* If $b = c$, then this is immediate by considering the $W(A)$-linear exact sequence $M_{(a)} \to N_{(b)} \to P_{(d)}$. But any $(F^k, F^\ell)$-linear map is also $(F^{k+m}, F^{\ell+m})$-linear for all $m \geq 0$, so we can assume $b = c$ without loss of generality. □

To complete the proof of Theorem 5.3.4, we only need to check that all of the maps in our exact sequences were $(F^k, F^\ell)$-linear for some choices of $k$ and $\ell$. The $F$ and $R$ maps are respectively $(F^0, F^1)$-linear and $(F^0, F^0)$-linear, and $V^n$ is $(F^n, F^0)$-linear. The following lemma implies (TO DO: why?) that $d$ is $(F^n, F^n)$-linear, and that the map $(V^n, dV^n)$ is $(F^{2n}, F^n)$-linear. Lemma 5.5.2: Let $R$ be a commutative ring (for us, an $\mathbb{F}_p$-algebra). Let $M$

be a $W_n(R)$-module annihilated by some $p^k M$, and let $d : W_n(R) \to M$ be a derivation. Then $d \circ F^k : W_{n+k}(R) \to M$ vanishes.

*Proof.* An element of $W_{n+k}(R)$ can be written (in Witt coordinates) as a sum of elements of the form $V^m[a]$, where $[a]$ denotes a Teichmüller representative. So it suffices to prove that $dF^k$ vanishes on inputs $x = V^m[a]$. But there are formulas that imply $\operatorname{im} F^k \subseteq p^k W_n(R)$: if $m \geq k$, then $F^k(x) = p^k V^{m-k}([a])$; and if $m \leq k$, then $F^k x = p^m [a]^{p^{k-m}}$. In either case, we have $dF^k(x) \in p^k M = 0$. □

# 6 Aside: calculations

Let's calculate the saturated de Rham-Witt complex of the $\mathbb{F}_p$-algebra $R = \mathbb{F}_p[t^{\pm 1}]$. We begin by choosing the smooth lift $\widetilde{R} = \mathbb{Z}_p[t^{\pm 1}]$, which is equipped with the lift of Frobenius $\varphi$ that acts as the identity (really, Witt vector Frobenius) on $\mathbb{Z}_p$ and sends $t$ to $t^p$. The de Rham complex of $\widetilde{R}$ is

$$\Omega_{\widetilde{R}} = \left( \widetilde{R} \to \widetilde{R}\, \frac{dt}{t} \right). \tag{11}$$

The $p$-adic completion of this is:

$$\widehat{\Omega}_{\widetilde{R}} = \left( \mathbb{Z}_p\{t^{\pm 1}\} \to \mathbb{Z}_p\{t^{\pm 1}\}\, \frac{dt}{t} \right). \tag{12}$$

(Note that the $p$-adic completion contains some elements such as $(1-pt)^{-1} = 1 + pt + p^2t^2 + \cdots$ that are not in $\widetilde{R}$.) This is a Dieudonné algebra, where $F$ acts as (the $p$-adic completion of) $\varphi$ on $\widehat{\Omega}^0_{\widetilde{R}}$ and likewise on the corresponding multiples of $\frac{dt}{t} \in \widehat{\Omega}^1_{\widetilde{R}}$. (Recall that $F$ is the divided Frobenius, so for example $F(dt) = t^{p-1}dt$, not $pt^{p-1}dt$.)

Next, we take the saturation of $\widehat{\Omega}^*_{\widetilde{R}}$. This lives inside $\widehat{\Omega}^*_{\widetilde{R}}[F^{-1}]$ (not a complex—e.g. it doesn't contain $d(t^{1/p})$), which looks like

$$\widehat{\Omega}^0_{\widetilde{R}}[F^{-1}] = \lim_{n \to} \mathbb{Z}_p\{t^{\pm 1/p^n}\} \tag{13}$$

$$\widehat{\Omega}^1_{\widetilde{R}}[F^{-1}] = \lim_{n \to} \mathbb{Z}_p\{t^{\pm 1/p^n}\}\, \frac{dt}{t}. \tag{14}$$

(A word of caution: the direct limit of $\mathbb{Z}_p\{t^{\pm 1/p^n}\}$ isn't quite what you might hope: its elements can be infinite series, but must have bounded denominators in their exponents.) An element $x$ in $\widehat{\Omega}^i_{\widetilde{R}}[F^{-1}]$ belongs to $\operatorname{Sat} \widehat{\Omega}^i_{\widetilde{R}}$ if and only if $dF^n x \in p^n \widehat{\Omega}^{i+1}_{\widetilde{R}}$ for sufficiently large $n$. This condition is of course vacuous for $x$ in degree 1, but for $x$ in degree 0 it says that every monomial in the power series $x$ must have the form $ct^\alpha$, where $c\alpha \in \mathbb{Z}_p$. So for example we allow $p^n t^{1/p^n}$ for all $n \geq 0$, but not $p^{n-1} t^{1/p^n}$. So $\operatorname{Sat} \widehat{\Omega}^*_{\widetilde{R}}$ looks like

$$\operatorname{Sat} \widehat{\Omega}^*_{\widetilde{R}} = \left( \lim_{n \to} \mathbb{Z}_p\{p^m t^{\pm a/p^m} : 0 \leq m \leq n\} \to \lim_{n \to} \mathbb{Z}_p\{t^{\pm 1/p^n}\}\, \frac{dt}{t} \right). \tag{15}$$

Again, the degree-0 piece of this only includes elements with bounded denominators in their exponents, and with coefficients converging $p$-adically to 0.

Next we take the strictification. The Frobenius acts as $t^a \mapsto t^{pa}$ and $t^a dt/t \mapsto t^{pa}\frac{dt}{t}$, so the Verschiebung acts as $t^a \mapsto pt^{a/p}$ and $t^a dt/t \mapsto pt^{a/p}\frac{dt}{t}$. It follows that an element in $\operatorname{Sat}\widehat{\Omega}^*_{\widetilde{R}}$ is in the image of $V^n$ if and only if all its coefficients are divisible by $p^n$, and an element in $\operatorname{Sat}\widehat{\Omega}^1_{\widetilde{R}}$ is in the image of $dV^n$ if and only if it can be integrated to such a function. The latter property says exactly that there is no $\frac{dt}{t}$ term and that each $ct^\alpha\frac{dt}{t}$ term (for $0 \neq \alpha \in \mathbb{Z}[1/p]$) satisfies $\frac{c}{\alpha} \in p^n\,\mathbb{Z}_p$. Note that the condition $\frac{c}{\alpha} \in p^n\,\mathbb{Z}_p$ is automatic when $\alpha$ has sufficiently many $p$'s in its denominator.

So the saturated de Rham-Witt complex of $\mathbb{F}_p[t^{\pm 1}]$ is as follows:

$$\mathcal{W}\Omega^*_R = \mathcal{W}\operatorname{Sat}\widehat{\Omega}^*_{\widetilde{R}} = \left(\left(\bigoplus_{\alpha\in\mathbb{Z}[1/p]} \max(|\alpha|_p, 1)\,\mathbb{Z}_p\,t^\alpha\right)^\wedge \to \left(\bigoplus_{\alpha\in\mathbb{Z}[1/p]} \mathbb{Z}_p\,t^\alpha\,\frac{dt}{t}\right)^\wedge\right), \tag{16}$$

where the completion in degree 0 allows infinite sums when the coefficients converge $p$-adically to 0 (this is different from the literal $p$-adic completion!), and the completion in degree 1 allows infinite sums of $ct^\alpha\frac{dt}{t}$ when the numbers $\frac{c}{\alpha} \in \mathbb{Q}_p$ converge $p$-adically to 0.

For $\mathbb{A}^1$, all of the discussion above goes through with very slight changes. The result is that in degree 0, we only get power series involving nonnegative powers of $t$, and in degree 0, we only get power series involving positive powers of $t$ times $\frac{dt}{t}$. The natural map from $\mathcal{W}\Omega^*_{k[t]}$ to $\mathcal{W}\Omega^*_{k[t^{\pm1}]}$ is the obvious one.

Finally, let's understand the saturated de Rham-Witt complex of $\mathbb{P}^1$ and how to compute crystalline cohomology from it.

Idea: $H^i_{\mathrm{cris}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1/W_n(\mathbb{F}_p)}) = \mathbb{H}^i(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1})$. Write down $\mathcal{W}_n\Omega^*$ for $\mathbb{G}_m$ and for $\mathbb{A}^1$, and calculate their cohomology. The cohomology sheaves $\mathcal{H}^j(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1})$ are glued from this data, and there is a spectral sequence

$$H^i(\mathbb{P}^1, \mathcal{H}^j(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1})) \implies \mathbb{H}^{i+j}(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1}). \tag{17}$$

The sheaf cohomology can be calculated as Čech cohomology of quasicoherent sheaves on the finite-level Witt scheme $W_n(\mathbb{P}^1)$, which we can do directly from what we know by now. This gives

$$H^0(\mathbb{P}^1, \mathcal{H}^0(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1})) = \mathbb{Z}/p^n\mathbb{Z}, \tag{18}$$

$$H^1(\mathbb{P}^1, \mathcal{H}^1(\mathbb{P}^1, \mathcal{W}_n\Omega^*_{\mathbb{P}^1})) = \mathbb{Z}/p^n\mathbb{Z}\cdot\frac{dt}{t}, \tag{19}$$

and the other terms are 0. So the spectral sequence degenerates, and we have

$$H^0_{\mathrm{cris}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1/W_n(\mathbb{F}_p)}) \cong H^2_{\mathrm{cris}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1/W_n(\mathbb{F}_p)}) \cong \mathbb{Z}/p^n\mathbb{Z}. \tag{20}$$

Taking inverse limits gives

$$H^0_{\mathrm{cris}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1/\mathbb{Z}_p}) \cong H^2_{\mathrm{cris}}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1/\mathbb{Z}_p}) \cong \mathbb{Z}_p. \tag{21}$$